About The Second Neighborhood Problem in Tournaments Missing Disjoint Stars

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Abstract

Let D be a digraph without digons. Seymour's second neighborhood conjecture states that D has a vertex v such that $d^+(v) \leq d^{++}(v)$. Under some conditions, we prove this conjecture for digraphs missing n disjoint stars. Weaker conditions are required when $n = 2$ or 3. In some cases we exhibit 2 such vertices.

1 Introduction

Let D be a digraph without digons (directed cycles of length 2). $V(D)$ and $E(D)$ denote its vertex set and edge set respectively. If $K \subseteq V(D)$ then the induced restriction of D to K is denoted by $D[K]$. As usual, $N^+_D(v)$ (resp. $N^-_D(v)$) denotes the (first) out-neighborhood (resp. in-neighborhood) of a vertex $v \in V$. $N^{++}_D(v)$ (resp. $N^{--}_D(v)$) denotes the second out-neighborhood (in-neighborhood) of v, which is the set of vertices that are at distance 2 from v (resp. to v). We also denote $d^+_D(v) = |N^+_D(v)|$, $d^{++}_D(v) = |N^{++}_D(v)|$, $d^-_D(v) = |N^-_D(v)|$ and $d^{--}_D(v) = |N^{--}_D(v)|$. We omit the subscript if the digraph is clear from the context. The minimum out-degree and the minimum in-degree of D, are denoted by δ^+_D and δ^-_D respectively. For short, we write $x \to y$ if the arc $(x, y) \in E$. A vertex $v \in V(D)$ is called whole if $d(v) := d^+(v) + d^-(v) = |V(D)| - 1$, otherwise v is non whole. A sink is a vertex of zero out-degree. For $x, y \in V(D)$, we say xy is a missing edge of D if neither (x, y) nor (y, x) are in $E(D)$. The missing graph G of D is the graph whose edges are the missing edges of D and whose vertices are the non whole vertices of D. In this case, we say that D is missing G. So, a tournament does not have missing edges. A tournament T is said to be a *completion* of a digraph D if $V(T) = V(D)$ and $E(D) \subseteq E(T)$, *i.e.* T is a tournament obtained from D by adding missing arcs.

A vertex v of D is said to have the second neighborhood property (SNP) if $d^+(v) \leq d^{++}(v)$. Dean [1] conjectured that every tournament has a vertex with

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the SNP. Seymour conjectured a more general statement [1].

Conjecture 1. ***(Seymour's Second Neighborhood Conjecture (SNC))**[1]* *Every digraph has a vertex with the SNP.*

In 1996, Fisher [2] solved Dean's conjecture, thus asserting the SNC for tournaments. Fisher's proof uses a certain probability distribution on the vertices. Another proof of Dean's conjecture was given in 2000 by Havet and thomassé [3]. Their proof uses a tool called median orders. Furthermore, they proved that if a tournament has no dominated vertex then there are at least two vertices with the SNP.

Let $D = (V, E)$ be a digraph (vertex) weighted by a non-negative real valued function $\omega : V \to \mathcal{R}_+$. The weight of an arc (x, y) is the weight of its head y. The weight of a set of vertices (resp. edges) is the sum of the weights of its members. We say that a vertex v has the weighted SNP if $\omega(N^+(v)) \leq \omega(N^{++}(v))$. It is known that the SNC is equivalent to its weighted version: *Every weighted digraph has a vertex with the weighted SNP.*

A weighted median order $L = v_1v_2...v_n$ of a weighted digraph (D, ω) is an order of the vertices of D the maximizes the weight of the set of forward arcs of D, i.e., the set $\{(v_i, v_j) \in E; i < j\}$. In fact, L satisfies the feedback property: For all $1 \leq i \leq j \leq n$:

$$\omega(N^+_{[i,j]}(v_i)) \geq \omega(N^-_{[i,j]}(v_i))$$

and

$$\omega(N^-_{[i,j]}(v_j)) \geq \omega(N^+_{[i,j]}(v_j))$$

where $[i, j] := D[v_i, v_{i+1}, ..., v_j]$.

An order $L = v_1v_2...v_n$ satisfying the feedback property is called weighted local median order. When $\omega = 1$, we obtain the defintion of (local) median orders of a digraph ([3], [4]). The last vertex v_n of a weighted local median order $L = v_1v_2...v_n$ of (D, ω) is called a *feed* vertex of the weighted digraph (D, ω).

Let $L = v_1v_2...v_n$ be a weighted local median order. Among the vertices not in $N^+(v_n)$ two types are distinguished: A vertex v_j is good if there is $i \leq j$ such that $v_n \to v_i \to v_j$, otherwise v_j is a bad vertex. The set of good vertices of L is denoted by G^D_L (or G_L if there is no confusion). Clearly, $G_L \subseteq N^{++}(v_n)$. The last vertex v_n is called a feed vertex of L (local median order) [3].

A matching is a set of pairwise indepndent edges (i.e. having no vertex in common).A star is a graph (or digraph) which consists of edges (arcs) sharing exactly one specified vertex as an endpoint called the center. We say that n stars are disjoint if there's vertex sets are pairwise disjoint.

In [5], Ghazal, also used the notion of weighted median order to prove the weighted SNC for digraphs missing a generalized star. As a corollary, the

weighted version holds for digraphs missing a star, complete graph or a sun.

In 2007, Fidler and Yuster [4] proved that SNC holds for digraphs with minimum degree $|V(D)|$-2 (i.e. digraphs missing a matching), and tournaments minus a subtournament, using also the notion of median orders. They have also used another tool called dependency digraph. We will give a more general definition of these digraphs.

we say that a missing edge x_1y_1 loses to a missing edge x_2y_2 if: $x_1 \to x_2$, $y_2 \notin N^+(x_1) \cup N^{++}(x_1)$, $y_1 \to y_2$ and $x_2 \notin N^+(y_1) \cup N^{++}(y_1)$. We define the dependency digraph Δ of D as follows: Its vertex set consists of all the missing edges and $(ab, cd) \in E(\Delta)$ if ab loses to cd. Note that Δ may contain digons.

Definition 1. *[5] A missing edge ab is called* good *if:*
(i) $(\forall v \in V\backslash\{a, b\})[(v \to a) \Rightarrow (b \in N^+(v) \cup N^{++}(v))]$ *or*
(ii) $(\forall v \in V\backslash\{a, b\})[(v \to b) \Rightarrow (a \in N^+(v) \cup N^{++}(v))]$.
If ab satisfies (i) we say that (a, b) is a convenient orientation of ab.
If ab satisfies (ii) we say that (b, a) is a convenient orientation of ab.

Clearly, a missing edge ab is good if and only if its in-degree in Δ is zero.

2 Preliminary Lemmas

We will need the following results.

Theorem 1. *[3] Let $L = x_1 \cdots x_n$ be a median order of a tournament T. Then x_n has the SNP. Moreover, if T has no sink then it has at least 2 vertices with SNP.*

Let D be a digraph and let Δ denote its dependency digraph. Let C be a connected component of Δ. Set $K(C) = \{u \in V(D)$; there is a vertex v of D such that uv is a missing edge and belongs to C $\}$. The *interval graph of D*, denoted by $\mathcal{I}_D$ is defined as follows. Its vertex set consists of the connected components of Δ and two vertices C_1 and C_2 are adjacent if $K(C_1) \cap K(C_2) \neq \phi$. So $\mathcal{I}_D$ is the intersection graph of the family $\{K(C); C$ is a connected component of Δ $\}$. Let ξ be a connected component of $\mathcal{I}_D$. We set $K(\xi) = \cup_{C \in \xi} K(C)$. Clearly, if uv is a missing edge in D then there is a unique connected component ξ of $\mathcal{I}_D$ such that u and v belongs to $K(\xi)$. If $f \in V(D)$, we set $J(f) = \{f\}$ if f is a whole vertex, otherwise $J(f) = K(\xi)$, where ξ is the unique connected component of $\mathcal{I}_D$ such that $f \in K(\xi)$. Clearly, if $x \in J(f)$ then $J(f) = J(x)$ and if $x \notin J(f)$ then x is adjacent to every vertex in $J(f)$.

Let $L = x_1 \cdots x_n$ be a (weighted) (local) median order of a digraph D. For $i < j$, the set $[i, j] := [x_i, x_j] := \{x_i, x_{i+1}, ..., x_j\}$ is said to be an interval of L. We say that $K \subseteq V(D)$ is an interval of D if for every $u, v \in K$ we have: $N^+(u)\backslash K = N^+(v)\backslash K$ and $N^-(u)\backslash K = N^-(v)\backslash K$. Clearly, there is a

(weighted) (local) median order L of D such that every interval of D is again an interval of L. We say that D is good if the sets $K(\xi)$'s are intervals of D. Clearly, every good digraph has a (weighted) (local) median order L such that the $K(\xi)$'s form intervals of L. Such an order is called a good (weighted) (local) median order of the good digraph D.

Following the proof of theorem 1 in [3], we prove this lemma:

Lemma 1. *Let (D, ω) be a good weighted digraph and let L be a good weighted local median order of (D, ω), with feed vertex say f. Then for every $x \in J(f)$, $\omega(N^+(x)\backslash J(f)|) \leq \omega(G_L\backslash J(f))$. So if x has the weighted SNP in $D[J(f)]$, then it has the weighted SNP in D.*

Proof. The proof is by induction n the number of vertices of D. It is trivial for $n = 1$. Let $L = x_1, ..., x_n$ be a good weighted local median order of (D, ω). Since $J(f)$ is an interval of D, we may assume that $J(x_n) = \{x_n\}$. If L does not have any bad vertex then $N^-(x_n) = G_L$. Whence, $\omega(N^+(x_n)) \leq \omega(N^-(x_n)) = \omega(G_L)$ where the inequality is by the feedback property. Now suppose that L has a bad vertex and let i be the smallest such that x_i is bad. Since $J(x_i)$ is an interval of D and L, then every vertex in $J(x_i)$ is bad and thus $J(x_i) = [x_i, x_p]$ for some $p < n$. For $j < i$, x_j is either an out-neighbor of x_n or a good vertex, by definition of i. Moreover, if $x_j \in N^+(x_n)$ then $x_j \in N^+(x_i)$. So $N^+(x_n) \cap [1, i] \subseteq N^+(x_i) \cap [1, i]$. Equivalently, $N^-(x_i) \cap [1, i] \subseteq G_L \cap [1, i]$. Therefore, $\omega(N^+(x_n) \cap [1, i]) \leq \omega(N^+(x_i) \cap [1, i]) \leq \omega(N^-(x_i) \cap [1, i]) \leq \omega(G_L \cap [1, i])$, where the second inequality is by the feedback property. Now $L' = x_{p+1}, ..., x_n$ is good also. By induction, $\omega(N^+(x_n) \cap [p+1, n]) \leq \omega(G_{L'})$. Note that $G_{L'} \subseteq G_L \cap [p+1, n]$. Whence $\omega(N^+(x_n)) = \omega(N^+(x_n) \cap [1, i]) + \omega(N^+(x_n) \cap [p+1, n]) \leq \omega(G_L \cap [1, i]) + \omega(G_L \cap [p+1, n]) = \omega(G_L)$. The second part of the statement is obvious. □

Let L be a good weighted median order of a good digraph D and let f denote its feed vertex. We have for every $x \in J(f)$, $\omega(N^+(x)\backslash J(f)) \leq \omega(G_L\backslash J(f))$. Let $b_1, \cdots, b_r$ denote the bad vertices of L not in $J(f)$ and $v_1, \cdots, v_s$ denote the non bad vertices of L not in $J(f)$, both enumerated in increasing order with respect to their index in L.
If $\omega(N^+(f)\backslash J(f)) < \omega(G_L\backslash J(f))$, we set $Sed(L) = L$. If $\omega(N^+(f)\backslash J(f)) = \omega(G_L\backslash J(f))$, we set $sed(L) = b_1 \cdots b_r J(f) v_1 \cdots v_s$.

Lemma 2. *Let L be a good weighted median order of a good weighted digraph (D, ω). Then $Sed(L)$ is a good weighted median order of (D, ω).*

Proof. Let $L = x_1, ..., x_n$ be a good weighted local median order of (D, ω). If $Sed(L) = L$, there is nothing to prove. Otherwise, we may assume that $\omega(N^+(x_n)\backslash J(x_n)) = \omega(G_L\backslash J(x_n))$. The proof is by induction on r the number of bad vertices not in $J(x_n)$. Set $J(x_n) = [x_t, x_n]$. If $r = 0$, then for every $x \in J(x_n)$ we have $N^-(x)\backslash J(x_n) = G_L\backslash J(x_n)$. Whence, $\omega(N^+(x)\backslash J(x_n)) = \omega(G_L\backslash J(x_n)) = \omega(N^-(x)\backslash J(x_n))$. Thus, $Sed(L) = J(x_n)x_1...x_{t-1}$ is a good weighted median order. Now suppose $r > 0$ and let i be the smallest such that

$x_i \notin J(x_n)$ and is bad. As before, $J(x_i) = [x_i, x_p]$ for some $p < n$, $\omega(N^+(x_n) \cap [1,i]) \leq \omega(N^+(x_i) \cap [1,i]) \leq \omega(N^-(x_i) \cap [1,i]) \leq \omega(G_L \cap [1,i])$ and $\omega(N^+(x_n) \cap [p+1, t-1]) \leq \omega(G_L \cap [p+1, t-1])$. However, $\omega(N^+(x_n) \backslash J(x_n)) = \omega(G_L \backslash J(x_n))$, then the prvious inequalities are equalities. In particular, $\omega(N^+(x_i) \cap [1,i]) = \omega(N^-(x_i) \cap [1,i])$. Since $J(x_i)$ is an interval of L and D, then for every $x \in J(x_i)$ we have $\omega(N^+(x) \cap [1,i]) = \omega(N^-(x) \cap [1,i])$. Thus $J(x_i)x_1...x_{i-1}x_{p+1}...x_n$ is a good weighted median order. To conclude, apply the induction hypothesis to the good weighted median order $x_1...x_{i-1}x_{p+1}...x_n$.

□

Define now inductively $Sed^0(L) = L$ and $Sed^{q+1}(L) = Sed(Sed^q(L))$. If the process reaches a rank q such that $Sed^q(L) = y_1...y_n$ and $\omega(N^+(y_n) \backslash J(y_n)) < \omega(G_{Sed^q(L)} \backslash J(y_n))$, call the order L stable. Otherwise call L periodic.

A digraph is said to be non trivial if it has at least one arc.

Lemma 3. *Let D be a digraph missing disjoint stars such that the connected components of its dependency digraph are non trivial strongly connected. Then D is a good digraph.*

Proof. Let ξ be a connected component of $\mathcal{I}_D$. Assume first that $K(\xi) = K(C)$ for some directed cycle C of Δ, say $C = (a_1b_1, ..., a_nb_n)$, namely $a_i \to a_{i+1}$ and $b_{i+1} \notin N^+(a_i) \cup N^{++}(a_i)$. If the set of edges $\{a_ib_i\}_i$ forms a matching then by lemma 3.3 in [4], we have the desired result. So, we will suppose that a center x of a missing star appears twice in the list $a_1, b_1, ..., a_n, b_n$ and assume without loss of generality that $x = a_1$. Suppose n is even. Set $K_1 = \{a_1, b_2, ..., a_{n-1}, b_n\}$ and $K_2 = K(C) \backslash K_1$. Suppose that $a_n \to b_1$ and $a_1 \notin N^+(a_n) \cup N^{++}(a_n)$. Then by following the proof of lemma 3.3 in [4] we obtain the desired result. Suppose $a_n \to a_1$ and $b_1 \notin N^+(a_n) \cup N^{++}(a_n)$. By using the same argument of lemma 3.3 in [4], we have that K_1 and K_2 are intervals of D. Assume, for contradiction, that $K_1 \cap K_2 = \phi$ and let $i > 1$ be the smallest index for which x is incident to a_ib_i. Clearly $i > 2$. However, $b_3 \notin K_1$ and $x = a_1 \to a_2 \to a_3$ implies that $i > 3$. Suppose that $x = a_i$. Since $b_2 \to a_1 = x = a_i$ and $a_3 \notin N^+(b_2) \cup N^{++}(b_2)$ then $a_3 \to x$. Similarly $b_4, a_5, ..., b_{i-1}$ are in-neighbors of x. However, b_{i-1} is an out-neighbor of $a_i = x$, a contradiction. Suppose that $x = b_i$. Similarly, $a_3, b_4, ..., a_{i-1}$ are in-neighbors of x. However, a_{i-1} is an out-neighbor of x, a contradiction. Thus $K_1 \cap K_2 \neq \phi$. whence, the desired result follows. Similar argument is used to prove it when C is an odd directed cycle.
This result can be easily extended to the case when $K(\xi) = K(C)$ and C is a non trivial (having more than one vertex) strongly connected component of Δ, because between any two missing egdes uv and zt there is directed path from uv to zt and a directed path from zt to uv. These two directed paths will form many directed cycles that are used to prove the desired result. This also is extended to the case when $K(\xi) = \cup_{C \in \xi} K(C)$: Let u, u' be 2 vertices in $K(\xi)$. There is a non trivial strongly connected components C and C' containing u and u' respectively. Since ξ is a connected component of $\mathcal{I}_D$, there is a directed path $C = C_0, C_1, ..., C_n = C'$. For all $i > 0$, there is $u_i \in K(C_{i-1}) \cap K(C_n)$.

Therefore, we have: $N^+(u)\backslash K(\xi) = N^+(u_1)\backslash K(\xi) = ... = N^+(u_i)\backslash K(\xi) = ... = N^+(u_n)\backslash K(\xi) = N^+(u')\backslash K(\xi)$ and $N^-(u)\backslash K(\xi) = N^-(u_1)\backslash K(\xi) = ... = N^-(u_i)\backslash K(\xi) = ... = N^-(u_n)\backslash K(\xi) = N^-(u')\backslash K(\xi)$.

□

3 Main results

3.1 Removing n disjoint stars

We recall that a vertex x in a tournament T is a king if $\{x\} \cup N^+(x) \cup N^{++}(x) = V(T)$. It is well known that every tournament has a king. However, for every natural number $n \notin \{2, 4\}$, there is a tournament T_n on n vertices, such that every vertex is a king for this tournament.

Theorem 2. *Let D be a digraph obtained from a tournament by deleting the edges of disjoint stars. Suppose that, in the induced tournament by the centers of the missing stars, every vertex is a king. If $\delta^-_\Delta > 0$ then D satisfies SNC.*

Proof. Orient all the missing edges towards the centers of the missing stars. Let L be a median order of the obtained tournament T and let f denote its feed vertex. We have $d^+_T(f) \leq d^{++}_T(f)$. It is easy to prove that if f is a whole vertex, then it has the SNP in D.

Suppose that f is the center of a missing star. In this case $N^+(f) = N^+_T(f)$. Suppose $f \to u \to v$ in T. If $(u, v) \in D$ then $v \in N^+(f) \cup N^{++}(f)$. Otherwise, uv is a missing edge, hence v is the center of a missing star, whence $v \in N^+(f) \cup N^{++}(f)$ because f is a king for the centers of the missing stars. Thus $N^{++}(f) = N^{++}_T(f)$. Therefore f has the SNP in D.
Now suppose that fx is a missing edge belonging to some missing star of center x. Suppose, first, that fx loses to a missing edge by, say y is the center of the missing star containing by. Assume $f \to x \to q$ in T with $q \neq y$, then $b \to y$, whence, $f \to b \to q$. Assume that $f \to c \to z$ in T, for some missing edge cz with $z \neq y$. Since $\delta^-_\Delta > 0$ there is a missing edge uv, with $x \notin \{u, v\}$ that loses to cz, namely, $v \to z$ and $c \notin N^+(v) \cup N^{++}(v)$. But $f \to c$ then $f \to v$, hence $f \to v \to z$ and $z \in N^+(f) \cup N^{++}(f)$. Thus y is the only new second out-neighbor of f. Note that f have lost x as a second out-neighbor and became a first out-neighbor. Therefore, $d^+(f) + 1 = d^+_T(f) \leq d^{++}_T(f) = d^{++}(f)$.
Suppose that fx does not lose to any edge. Reorient fx from x to f. The same order L is a median order for the new tournament T' and $N^+(f) = N^+_{T'}(f)$. Suppose that $f \to c \to z$ with cz is a missing edge and $z \notin N^+(f) \cup N^{++}(f)$. Assume that ax is a missing edge that loses to cz. Then $x \to z$ and $c \notin N^+(z) \cup N^{++}(z)$. Whence, fx loses to cz, a contradiction. Since $\delta^-_\Delta > 0$ there is a missing edge by, with $x \notin \{b, y\}$ that loses to cz, namely, $y \to z$ and $c \notin N^+(y) \cup N^{++}(y)$. But $f \to c$ then $f \to y$, hence $f \to y \to z$ and $z \in N^+(f) \cup N^{++}(f)$. Thus, $N^{++}(f) = N^{++}_{T'}(f)$. Therefore, f has the SNP in D. □

Theorem 3. *Let D be a digraph whose missing graph is disjoint union of one star and a matching. If every connected component of the dependency digraph containing an edge of the missing star, has positive minimum out-degree and positive minimum in-degree, then D satisfies SNC.*

Let D be a digraph such that its missing graph is disjoint union of a star S_x of center x and a matching M. Δ and $\mathcal{I}_D$ denote the dependency digraph and the interval graph of D respectively. In addition, we suppose that each connected component of Δ containing a missing edge of D incident to x (edge of the missing star) has positive minimum . out-degree and positive minimum in-degree. In what follows, we prove that D satisfies SNC.

Let P be a connected component of Δ or $\mathcal{I}_D$ and let v be a vertex of D. We say that v appears in P if $v \in K(P)$. Otherwise, we say v does not appear in P.

Note that we can use the same argument of lemma 3.1 in [4] to prove that the in-degree and out-degree in Δ of every edge ax of the missing star S_x is exactly one, and that if an edge uv of M has out-degree (resp. in-degree) more than one then $N^+_\Delta(uv) \subseteq E(S_x)$ (resp, $N^-_\Delta(uv) \subseteq E(S_x)$). So every connected component of Δ, in which x does not appear, is either a directed path or a directed cycle.

We denote by ξ the unique connected component of $\mathcal{I}_D$ in which x appears. So $\mathcal{I}_D$ is composed of the connected componet ξ and other isolated vertices.

Let $P = a_1b_1, a_2b_2, \cdots, a_kb_k$ be a connected component of Δ, which is also a maximal path in Δ in which x does not appear, namely $a_i \to a_{i+1}, b_i \to b_{i+1}$ for $i = 1, ..., k-1$. Since a_1b_1 is a good edge then (a_1, b_1) or (b_1, a_1) is a convenient orientation. If (a_1, b_1) is a convenient orientation , we orient (a_i, b_i) for $i = 1, ..., k$. Otherwise we orient a_ib_i as (b_i, a_i). We do this for all the components of Δ which are paths. Denote the set of these new arcs by F. Let $D' = D + F$, i.e., D' is obtained from D by adding the arcs in F.

Let ξ denote the unique connected component of $\mathcal{I}_D$ such that $x \in K(\xi)$.

Lemma 4. *D' is a good digraph.*

Proof. Lemma 3.3 in [4] proves that every set $K(C)$ is an interval of D whenever C is a directed cycle of Δ in which x does not appear.

Now we prove for all $u \in K(\xi)$, we have $N^+(u)\backslash K(\xi) = N^+(x)\backslash K(\xi)$. Let $u \in K(\xi)$ and let C denote the connected component of Δ in which u appears. Note that also x appears in C. If u appears in a non trivial strongly connected component then by the proof of lemma 3 the result follows. Otherwise, due to the condition that C has positive minimum out-degree and positive minimum in-degree, there is a directed path $P = u_1v_1, ..., u_kv_k$ joining two non trivial strongly connected components C_1 and C_2 contained in C such that u appears in P. The vertex x must appear in C_1 and C_2. By the proof of lemma 3, for

all $a \in K(C_1) \cup K(C_2)$, we have $N^+(a)\backslash K(\xi) = N^+(x)\backslash K(\xi)$. Due to the definition of losing relations between missing edges, we can easily show that for all $a \in K(C_1)$, $b \in K(P)$ and $c \in K(C_2)$ we have $N^+(a)\backslash K(\xi) \subseteq N^+(b)\backslash K(\xi) \subseteq N^+(c)\backslash K(\xi)$, in particular, for $a = x = c$ and $b = u$. So $K(\xi)$ is an interval of D.

This shows also that the depndency digraph $\Delta^{'}$ of D' is obtained from Δ by deleting the components that are directed paths not containing x. So the above intervals of D are also intervals of D'. Whence D' is a good digraph.

□

Lemma 5. *D[K((ξ)] satisfies SNC.*

Proof. Set $A = V(S_x) - x$. For all $a \in A$, orient ax as (a, x). Let $uv \in M$ such that $u, v \in K(\xi)$. Let P be the shortest path in Δ starting with an edge of the star S_x and ending in uv, namely, $P = ax, u_1v_1, ..., u_nv_n$ with $x \rightarrow v_1$, $v_i \rightarrow v_{i+1}$ for all $i < n$ and $u_nv_n = uv$. We orient uv from u_n to v_n. We do this for all the missing edges of $D[K(\xi)]$. We denote the obtained tournament by $T[K(\xi)]$.

Let L be a median order of $T[K(\xi)]$ which maximizes α the index of x and let g denote its feed vertex. In addition to the fact that g has the SNP in $T[K(\xi)]$, g has the SNP in $D[K(\xi)]$. In fact, if $g = x$ then clearly g gains no out-neighbor. Moreover, g does not gain any new second out-neighbor. Suppose that $g \rightarrow u \rightarrow v \rightarrow g$, with $uv \in M$. Since $x \rightarrow u$ and uv is oriented from u to v, then for every $a \in A$, $ax \rightarrow uv$ in Δ, whence there is a missing edge $u'v'$ that loses to uv, say, $v' \rightarrow v$ and $u \notin N^{++}(v')$. But $x \rightarrow u$, then $x \rightarrow v'$, whence $x \rightarrow v' \rightarrow v$ in D. So x gains no new second out-neighbor, so it has the SNP in $D[K(\xi)]$ also. Suppose that $g = a \in A$. Then a gains only x in its first out-neighbor. There is a unique missing rs with $ax \rightarrow rs$, say $a \rightarrow r$ and $s \notin N^+(a) \cup N^{++}(a)$. Then $(r, s) \in T[K(\xi)]$. Suppose that $a \rightarrow u \rightarrow v \rightarrow a$ in $T[K(\xi)]$ with $uv \in M - rs$. There is a missing edge $u'v'$ that loses to uv, say, $v' \rightarrow v$ and $u \notin N^{++}(v')$. But $a \rightarrow u$, then $a \rightarrow v' \rightarrow v$. Suppose that $a \rightarrow x \rightarrow q$ in $T[K(\xi)]$ with $q \neq s$. Since $x \rightarrow q$ in D and $r \notin N^{++}(x)$ then $r \rightarrow q$, whence $a \rightarrow r \rightarrow q$ in $D[K(\xi)]$. Note that a loses x as second out-neighbor in $T[K(\xi)]$. We get $d^+_{D[K(\xi)]}(a) + 1 = d^+_{T[K(\xi)]}(a) \leq d^{++}_{T[K(\xi)]}(a) = d^{++}_{D[K(\xi)]}(a)$, whence, a has the SNP in $D[K(\xi)]$. Similar argument can be used in the case when g is incident to a missing edge of M, that is oriented out of g , to show g has the SNP in $D[K(\xi)]$. Suppose that g is incident to a missing edge of M, that is oriented towards g. We can use similar arguments as above, to show that x is the only possible new second out-neighbor of g. If $x \in G_L$ and $d^+_{T[K(\xi)]}(g) = |G_L|$ then $sed(L)$ is a median order of $T[K(\xi)]$, in which the index of x is greater than α, a contradiction. Otherwise, $x \notin G_L$ or $d^+_{T[K(\xi)]}(g) < |G_L|$, whence, $d^+_{D[K(\xi)]}(g) = d^+_{T[K(\xi)]}(g) \leq d^{++}_{D[K(\xi)]}(g)$, hence g has the SNP in $D[K(\xi)]$ in this case. So g has the SNP in $D[K(\xi)]$ and $D[K(\xi)]$ satisfies SNC. □

In the following, $C = a_1b_1, ..., a_kb_k$ denotes a directed cycle of Δ in which x does not appear, namely $a_i \rightarrow a_{i+1}$, $b_{i+1} \notin N^{++}(a_i) \cup N^+(a_i)$, $b_i \rightarrow b_{i+1}$ and

$a_{i+1} \notin N^{++}(b_i) \cup N^{+}(b_i)$. In [4], it is proved that $D[K(C)]$ satisfies SNC. Here we prove that every vertex of $K(C)$ has the SNP in $D[K(C)]$.

Lemma 6. *([4])If k is odd then $a_k \rightarrow a_1$, $b_1 \notin N^{++}(a_k)$, $b_k \rightarrow b_1$ and $a_1 \notin N^{++}(b_k)$. If k is even then $a_k \rightarrow b_1$, $a_1 \notin N^{++}(a_k)$, $b_k \rightarrow a_1$ and $b_1 \notin N^{++}(b_k)$.*

Lemma 7. *In $D[K(C)]$ we have:*
k is odd:

$$N^{+}(a_1) = N^{-}(b_1) = \{a_2, b_3, \cdots, a_{k-1}, b_k\}$$

$$N^{-}(a_1) = N^{+}(b_1) = \{b_2, a_3, \cdots, b_{k-1}, a_k\},$$

k is even:

$$N^{+}(a_1) = N^{-}(b_1) = \{a_2, b_3, \cdots, b_{k-1}, a_k\}$$

$$N^{-}(a_1) = N^{+}(b_1) = \{b_2, a_3, \cdots, a_{k-1}, b_k\}.$$

Proof. Suppose that k is odd. Since (a_k, a_1, b_k, b_1) is a losing cycle, then $b_k \in N^{+}_{D[K(C)]}(a_1)$. Since $(a_{k-1}, a_k, b_{k-1}, b_k)$ is a losing cycle and $(a_1, b_k) \in E(D)$ then $(a_1, a_{k-1}) \in E(D)$ and so $a_{k-1} \in N^{+}_{D[K(C)]}(a_1)$, since otherwise $(a_{k-1}, a_1) \in E(D)$ and so $b_k \in N^{++}_{D[K(C)]}(a_{k-1})$, contradiction to the definition of the losing cycle $(a_{k-1}, a_k, b_{k-1}, b_k)$. And so on $b_{k-2}, a_{k-3}, ..., b_3, a_2 \in N^{+}_{D[K(C)]}(a_1)$. Again, since (a_1, a_2, b_1, b_2) is a losing cycle then $b_2 \in N^{-}_{D[K(C)]}(a_1)$. Since (a_2, a_3, b_2, b_3) is a losing cycle and $(b_2, a_1) \in E(D)$ then $(a_3, a_1)\ inE(D)$ and so $a_3 \in N^{-}_{D[K(C)]}(a_1)$. And so on, $b_4, a_5, ..., b_{k-1}, a_k \in N^{-}_{D[K(C)]}(a_1)$. We use the same argument for finding $N^{+}_{D[k(C)]}(b_1)$ and $N^{-}_{D[k(C)]}(b_1)$. Also we use the same argument when k is even. □

Lemma 8. *In $D[K(C)]$ we have: $N^{+}(a_i) = N^{-}(b_i)$, $N^{-}(a_i) = N^{+}(b_i)$, $N^{++}(a_i) = N^{-}(a_i) \cup \{b_i\} \backslash \{b_{i+1}\}$ and $N^{++}(b_i) = N^{-}(b_i) \cup \{a_i\} \backslash \{a_{i+1}\}$ for all $i = 1, ..., k$ where $a_{k+1} := a_1$, $b_{k+1} := b_1$ if k is odd and $a_{k+1} := b_1$, $b_{k+1} := a_1$ if k is even. So $d^{++}(v) = d^{+}(v) = d^{-}(v) = k - 1$ for all $v \in K(C)$.*

Proof. The first part is due to the previous lemma and the symmetry in these cycles. For the second part it is enough to prove it for $i = 1$ and a_1. Suppose first that k is odd. By definition of losing relation between $a_1 b_1$ and $a_2 b_2$ we have $b_2 \notin N^{++}(a_1) \cup N^{+}(a_1)$. Moreover $a_1 \rightarrow a_2 \rightarrow b_1$, whence $b_1 \in N^{++}(a_1)$. Note that for $i = 1, ..., k-1$, $a_i \rightarrow a_{i+1}$ and $b_i \rightarrow b_{i+1}$. Combining this with the previous lemma we find that $N^{++}(a_1) = N^{-}(a_1) \cup \{b_1\} \backslash \{b_2\}$. Similar argument is used when k is even.

□

Proof of theorem 3: Let L be a good median order of the digraph D'. Let f denotes its feed vertex and $J(f)$ denotes the interval of f. By lemma 1, for every $y \in J(f)$ we have $|N^{+}_{D'} \backslash J(f)| \leq |G_L \backslash J(f)|$. By lemmas 5 and 8, there is $y \in J(f)$ with the SNP in $D[J(f)] = D'[J(f)]$. So y has the SNP in D'. We prove that y has the SNP in D. Assume first that y is not an endpoint of any new arc of F. Clearly, y gains no new first out-neighbor. Suppose

$y \to u \to v$ in D' with $(u,v) \notin D$. If (u,v) is a convenient orientation, then $v \in N^+(y) \cup N^{++}(y)$. Otherwise, there is a missing edge rs that loses to uv, namely $s \to v$ and $u \notin N^+(s) \cup N^{++}(s)$. But $y \to u$ then $y \to s$, whence, $y \to s \to v$. So y gains no new second out-neighbor and thus y has the SNP in D. Now assume that $(z,y) \in F$ for some z. Then in this case also y gains neither a new first out-neighbor nor a new second out-neighbor. Now assume that $(y,z) \in F$. If yz is that last vertex of the directed path in Δ then we reorient it as (z,y). The same L is a median order of D', however, y gains neither a new second out-neighbor nor a new first out-neighbor. the last case to consider is when $z = a_i$ and $(a_i, b_i) \in F$ and $a_i b_i \to a_{i+1} b_{i+1}$ inΔ. In this case, gains only b_i one first out-neighbor and only b_{i+1} as a new second out-neighbor. Thus, y has the SNP in D.

Corollary 1. *[4] Every digraph missing a matching satisfies SNC.*

We note that our method guarantees that the vertex found with the SNP is a feed vertex of some digraph containg D. This is not guaranteed by the proof presented in [4]. Recall that F is the set of the new arcs added to D to obtain the good digraph D'. So if $F = \phi$ then D is a good digraph.

Theorem 4. *Let D be a digraph missing a matching and suppose $F = \phi$. If D does not have a sink then it has 2 vertices with the SNP.*

Proof. Consider a good median order $L = x_1 ... x_n$ of D. If $J(x_n) = K$ then by lemma 1 and lemma 8 the result holds. Otherwise, x_n is a whole vertex (i.e. $J(x_n) = \{x_n\}$). By lemma 1, x_n has the SNP in D. So we need to find another vertex with SNP. Consider the good median order $L' = x_1 ... x_{n-1}$. Suppose first that L' is stable. There is q for which $Sed^q(L') = y_1 ... y_{n-1}$ and $|N^+(y_{n-1}) \backslash J(y_{n-1})| < |G_{Sed^q(L')} \backslash J(y_{n-1})|$. Note that $y_1 ... y_{n-1} x_n$ is also a good median order of D. By lemma 8 and lemma 1, $y := y_{n-1}$ has the SNP in $D[y_1, y_{n-1}]$. So $|N^+(y)| = |N^+_{D[y_1,y_{n-1}]}(y)| + 1 \leq |G_{Sed^q(L')}| \leq |N^{++}(y)|$. Now suppose that L' is periodic. Since D has no sink then x_n has an out-neighbor x_j. Note that for every q, x_n is an out-neighbor of the feed vertex of $Sed^q(L')$. So x_j is not the feed vertex of any $Sed^q(L')$. Since L' is periodic, x_j must be a bad vertex of $Sed^q(L')$ for some integer q, otherwise the index of x_j would always increase during the sedimentation process. Let q be such an integer. Set $Sed^q(L') = y_1 ... y_{n-1}$. Lemma 8 and lemma 1 guarantees that the vertex $y := y_{n-1}$ with the SNP in $D[y_1, y_{n-1}]$. Note that $y \to x_n \to x_j$ and $G_{Sed^q(L')} \cup \{x_j\} \subseteq N^{++}(y)$. So $|N^+(y)| = |N^+_{D[y_1,y_{n-1}]}(y)| + 1 = |G_{Sed^q(L')}| + 1 = |G_{Sed^q(L')} \cup \{x_j\}| \leq |N^{++}(y)|$.

□

3.2 Removing a star

A more general statement to the following theorem is proved in [5]. Here we give another prove that uses the sedimentation technique of a median order.

Theorem 5. *[5] Let D be a digraph obtained from a tournament by deleting the edges of a star. Then D satisfies SNC.*

Proof. Orient all the missing edges of D towards the center x of the missing star. The obtained digraph is a tournament T completing D. Let L be a median order of T that maximizes α the index of x in L and let f denote its feed vertex. If $f = x$ then, clearly, $d^+(f) = d^+_{T'}(f) \leq |G_L^{T'}| \leq d^{++}_{T'}(f) = d^{++}(f)$. Now suppose that $f \neq x$. Reorient the missing edges incident to f towards f (if any). L is also a median order of the new tournament T'. Note that $N^+(f) = N^+_{T'}(f)$ and we have $d^+_{T'}(f) \leq |G_L^{T'}|$. If $x \in G_L^{T'}$ and $d^+_{T'}(f) = |G_L^{T'}|$ then $sed(L)$ is a median order of T' in which the index of x is greater than α, and also greater than the index of f. So we can give the missing edge incident to f (if it exists it is xf) its initial orientation (as in T) such that $sed(L)$ is a median order of T, a contradiction to the fact that L maximizes α. So $x \notin G_L^{T'}$ or $d^+T'(f) < |G_L^{T'}|$. We have that x is the only possible gained second out-neighbor vertex for f. If $x \notin G_L^{T'}$ then $G_L^{T'} \subseteq N^{++}(f)$, whence the result follows. If $d^+_{T'}(f) < |G_L^{T'}|$ then $d^+(f) = d^+_{T'}(f) \leq |G_L^{T'}| - 1 \leq d^{++}(f)$. So f has the SNP in D. □

3.3 Removing 2 disjoint stars

In this section let D be a digraph obtained from a tournament by deleting the edges of 2 disjoint stars. Let S_x and S_y be the two missing disjoint stars with centers x and y respectively, $A = V(S_x)\backslash x$, $B = V(S_y)\backslash y$, $K = V(S_x) \cup V(S_y)$ and assume without loss of generality that $x \to y$. In [5] it is proved that if the dependency digraph of any digraph consists of isolated vertices only then it satisfies SNC. Here we consider the case when the dependency digraph of D has no isolated vertices.

Theorem 6. *Let D be a digraph obtained from a tournament by deleting the edges of 2 disjoint stars. If $\delta_\Delta > 0$, then D satisfies SNC.*

Proof. Assume without loss of generality that $x \to y$. We note that the condition $\delta_\Delta > 0$ implies that for every $a \in A$ and $y \in B$ we have $y \to a$ and $b \to x$. We shall orient the missing edges to obtain a completion of D. First, we give every good edge a convenient orientation. For the other missing edges, let the orientation be towards the center of the 2 missing stars S_x or S_y. The obtained digraph is a tournament T completing D. Let L be a median order of T such that the index k of x is maximum and let f denote its feed vertex. We know that f has the SNP in T. We have only 5 cases:
Suppose that f is a whole vertex. In this case $N^+(f) = N^+_T(f)$. Suppose $f \to u \to v$ in T. Clearly $(f,u) \in D$. If $(u,v) \in D$ or is a convenient orientation then $v \in N^+(f) \cup N^{++}(f)$. Otherwise there is a missing edge zt that loses to uv with $t \to v$ and $u \notin N^+(t) \cup N^{++}(t)$. But $f \to u$, then $f \to t$, whence $f \to t \to v$ in D. Therefore, $N^{++}(f) = N^{++}_T(f)$ and f has the SNP in D as well.
Suppose $f = x$. Orient all the edges of S_x towards the center x. L is a median order of the modified completion T' of D. We have $N^+(f) = N^+_{T'}(f)$. Suppose

$f \to u \to v$ in T'. If $(u,v) \in D$ or is a convenient orientation then $v \in N^+(f) \cup N^{++}(f)$. Otherwise $(u,v) = (b,y)$ for some $b \in B$, but $f = x \to y$. Thus, $N^{++}(f) = N^{++}_{T'}(f)$ and f has the SNP in T' and D.
Suppose $f = b \in B$. Orient the missing edge by towards b. Again, L is a median order of the modified tournament T' and $N^+(f) = N^+_{T'}(f)$. Suppose $f \to u \to v$ in T'. If $(u,v) \in D$ or is a convenient orientation then $v \in N^+(f) \cup N^{++}(f)$. Otherwise $(u,v) = (b',y)$ for some $b' \in B$ or $(u,v) = (a,x)$ for some $a \in A$, however $x, y \in N^{++}(f) \cup N^+(f)$ because $f = b \to x \to y$ in D. Thus, $N^{++}(f) = N^{++}_{T'}(f)$ and f has the SNP in T' and D.
Suppose $f = y$. Orient the missing edges towards y and let T' denote the new tournament. We note that $B \subseteq N^{++}(y) \cap N^{++}_{T'}(y)$ due to the condition $\delta_\Delta > 0$. Also, x is the only possible new second neighbor of y in T'. If $B \cup \{x\} \subsetneqq G_L$ or $d^+_{T'}(y) < d^{++}_{T'}(y)$, then $d^+(y) = d^+_{T'}(y) \leq d^{++}_{T'}(y) - 1 \leq d^{++}(y)$. Otherwise, $B \cup \{x\} \subseteq G_L$ and $d^+_{T'}(y) = |G_L|$. In this case we consider the median order $Sed(L)$ of T'. Now the feed vertex of $sed(L)$ is different from y, the index of x had increased, and the index of y became less than the index of any vertex of B which makes $Sed(L)$ a median order of T also, in which the index of x is greater than k, a contradiction.
Suppose $f = a \in A$. Orient the missing edge ax as (x,a) and let T' denote the new tournament. Note that y is the only possible new second out-neighbor of a in T' and not in D. Also $x \in N^{++}_T(a) \cap N^{++}(a)$. If $d^+_{T'}(a) < d^{++}_{T'}(a)$, then $d^+(a) = d^+_{T'}(a) \leq d^{++}_{T'}(a) - 1 \leq d^{++}(a)$, hence a has the SNP in D. Otherwise, $d^+_{T'}(a) = |G_L| = d^{++}_{T'}(a)$ and in particular $x \in G_L$. In this case we consider $sed(L)$ which is a median order of T'. Note that the feed vertex of $Sed(L)$ is different from a and the index of a is less than the index of x in the new order $Sed(L)$. Hence $Sed(L)$ is a median of T as well, in which the index of x is greater than k, a contradiction.
So in all cases f has the SNP in D. Therefore D satisfies SNC. □

Theorem 7. *Let D be a digraph obtained from a tournament by deleting the edges of 2 disjoint stars. If $\delta^+_\Delta > 0$, $\delta^-_\Delta > 0$ and D does not have any sink, then D has at least two vertices with the SNP.*

Proof. **Claim 1:** *Suppose $K = V(D)$. If $\delta_\Delta > 0$, then D has at least two vertices with the SNP.*
proof-claim 1: The condition $\delta_\Delta > 0$ implies that for every $a \in A$ and $b \in B$ we have $y \to a$ and $b \to x$. Clearly, $N^+(x) = \{y\}$, $N^+(y) = A$, $d^+(x) \leq 1 \leq |A| \leq d^{++}(x)$, thus x has the SNP. Let H be the tournament $D - \{x,y\}$. Then H has a vertex v with the SNP in H. If $v \in A$, then $d^+v = d^+_H(v) \leq d^{++}_H(v) = d^{++}(v)$. If $v \in B$, then $d^+(v) = d^+_H(v) + 1 \leq d^{++}_H(v) + 1 = d^{++}(v)$. Whence, v also has the SNP in D.
Claim 2: *D is a good digraph.*
proof-claim 2: Let $\mathcal{I}_\mathcal{D}$ be the interval graph of D. Let C_1 and C_2 be two distinct connected components of Δ. Then the centers x and y appear in each of the these two connected components, whence $K(C_1) \cap K(C_2) \neq \phi$. Therefore, $\mathcal{I}_\mathcal{D}$ is a connected graph (more precisely, it is a complete graph), having only

one connected component ξ. Then, $K = K(\xi)$.
So, if Δ is composed of non trivial strongly connected components, the result holds by lemma 3.
Due to the condition $\delta^+_\Delta > 0$ and $\delta^-_\Delta > 0$, Δ has a non trivial strongly connected component, hence $N^+(x)\backslash K = N^+(y)\backslash K$. Now let $v \in K$ and assume without loss of generality that xv is a missing edge. Due to the condition $\delta^+_\Delta > 0$ and $\delta^-_\Delta > 0$, we have that either xv belongs to a non trivial strongly connected component of Δ, and in this case $N^+(v)\backslash K = N^+(x)\backslash K = N^+(y)\backslash K$, or xv belongs to a directed path $P = xa_1, yb_1, \cdots, xa_p, yb_p$ joining 2 non trivial strongly connected components C_1 and C_2 with $xa_1 \in C_1$ and $yb_p \in C_2$. There is $i > 1$ such that $v = a_i$. $L = xa_{i-1}, yb_{i-1}, xa_i, yb_i$ is a path in Δ. By the definition of losing cycles we have $N^+(x)\backslash K \subseteq N^+(b_{i-1})\backslash K \subseteq N^+(a_i)\backslash K \subseteq N^+(y)\backslash K = N^+(x)\backslash K$. Hence $N^+(x)\backslash K = N^+(v)\backslash K$ for all $v \in K$. Since every vertex outside K is adjacent to every vertex in K we also have $N^-(x)\backslash K = N^-(v)\backslash K$ for all $v \in K$.

Now, consider a good median order $L = x_1...x_n$ of D. If $J(x_n) = K$ then by claim 1 and lemma 1 the result holds. Otherwise, x_n is a whole vertex (i.e. $J(x_n) = \{x_n\}$). By lemma 1, x_n has the SNP in D. So we need to find another vertex with SNP. Consider the good median order $L' = x_1...x_{n-1}$. Suppose first that L' is stable. There is q for which $Sed^q(L') = y_1...y_{n-1}$ and $|N^+(y_{n-1})\backslash J(y_{n-1})| < |G_{Sed^q(L')}\backslash J(y_{n-1})|$. Note that $y_1...y_{n-1}x_n$ is also a good median order of D. Claim 1 and lemma 1 guarantees the existance of a vertex y with the SNP in $D[y_1, y_{n-1}]$. Since $y_{n-1} \to x_n$ and $y \in J(y_{n-1})$ which is an interval of D, then $y \to x_n$. So $|N^+(y)| = |N^+_{D[y_1,y_{n-1}]}(y)| + 1 \leq |G_{Sed^q(L')}| \leq |N^{++}(y)|$. Now suppose that L' is periodic. Since D has no sink then x_n has an out-neighbor x_j. Note that for every q, x_n is an out-neighbor of the feed vertex of $Sed^q(L')$. So x_j is not the feed vertex of any $Sed^q(L')$. Since L' is periodic, x_j must be a bad vertex of $Sed^q(L')$ for some integer q, otherwise the index of x_j would always increase during the sedimentation process. Let q be such an integer. Set $Sed^q(L') = y_1...y_{n-1}$. Claim 1 and lemma 1 guarantees the existance of a vertex y with the SNP in $D[y_1, y_{n-1}]$. Since $y_{n-1} \to x_n$ and $y \in J(y_{n-1})$ which is an interval of D, then $y \to x_n \to x_j$. Note that $G_{Sed^q(L')} \cup \{x_j\} \subseteq N^{++}(y)$. So $|N^+(y)| = |N^+_{D[y_1,y_{n-1}]}(y)| + 1 = |G_{Sed^q(L')}| + 1 = |G_{Sed^q(L')} \cup \{x_j\}| \leq |N^{++}(y)|$.

□

3.4 Removing 3 disjoint stars

In this section, D is obtained from a tournament missing the edges of 3 disjoint stars S_x, S_y and S_z with centers x, y and z respectively. Set $A = V(S_x) - x$, $B = V(S_y) - x$, $C = V(S_z) - z$ and $K = A \cup B \cup C \cup \{x, y, z\}$. Let Δ denote the dependency digraph of D. The triangle induced by the vertices x, y and z is either a transitive triangle or a directed triangle.
We will deal with the case when this triangle is directed, and assume without

loss of generality that $x \to y \to z \to x$. This is a particular case of the case when the missing graph is a disjoint union of stars such that, in the induced tournament by the centers of the missing stars, every vertex is a king.

Theorem 8. *Let D be a digraph obtained from a tournament by deleting the edges of 3 disjoint stars whose centers form a directed triangle. If $\delta_\Delta > 0$, then D satisfies EC.*

Proof. **Claim :** *The only possible arcs in Δ have the forms $xa \to yb$ or $yb \to zc$ or $zc \to xa$, where $a \in A$, $b \in B$ and $c \in C$.*

proof-claim : xa can not lose to zc because $z \to x$ and $z \in N^{++}(x)$. Similarly yb can not lose to xa and zc can not lose to yb.

Orient the good missing edges in a convenient way and orient the other edges toward the centers. The obtained digraph T is a tournament. Let L be a median order of T such that the sum of the indices of x, y and z is maximum. Let f denote the feed vertex of L. Due to symmetry, we may assume that f is a whole vertex or $f = x$ or $f = a \in A$. Suppose f is a whole vertex. Clearly, $N^+(f) = N^+_T(f)$. Suppose $f \to u \to v$ in T. If $(u, v) \in E(D)$ or uv is a good missing edge then $v \in N^+(f) \cup N^{++}(f)$. Otherwise, there is missing edge rs that loses to uv with $r \to v$ and $u \notin N^{++}(r) \cup N^+(r)$. But $f \to u$, then $f \to r$, whence $f \to r \to v$ and $v \in N^+(f) \cup N^{++}(f)$. Thus, $N^{++}_T(f) = N^{++}(f)$ and f has the SNP in D.
Suppose $f = x$. Reorient all the missing edges incident to x toward x. In the new tournament T' we have $N^+(x) = N^+_{T'}(x)$. Since $y \in N^+(x)$ and $z \in N^{++}(x)$ we have that $N^{++}(x) = N^{++}_{T'}(x)$. Thus x has the SNP in D.
Suppose that $f = a \in A$. Reorient ax toward a. Suppose $a \to u \to v$ in the new tournament T' with $v \neq y$. If $(u, v) \in E(D)$ or uv is a good missing edge then $v \in N^+(a) \cup N^{++}(a)$. Otherwise, there is $b \in B$ and $c \in C$ such that $(u, v) = (c, z)$ and by loses to cz, then $f \to c$ implies that $a \to y$, but $y \to z$, whence $z \in N^{++}(a) \cup N^+(a)$. So the only possible new second out-neighbor of a is y, hence if $y \notin N^{++}_{T'}(a)$ then a has the SNP in D. Suppose $y \in N^{++}_{T'}(a)$. If $d^+_{T'}(a) < d^{++}_{T'}(a)$ then $d^+(a) = d^+_{T'}(a) \leq d^{++}_{T'}(a) = d^{++}_{(}a)$, hence a has the SNP in D. Otherwise, $d^+_{T'}(a) = |G_L|$ and $G_L = N^{++}_{T'}(a)$. So x, y and z are not bad vertices, hence the index of each increases in the median order $Sed(L)$ of T'. But the index of a is less than the index of x, then we can give ax its initial orientation as in T and the same order $Sed(L)$ is a median order of T. However, the sum of indices of x, y and z have increased. A contradiction. Thus f has the SNP in D and D satisfies SNC. $\square$

Theorem 9. *Let D be a digraph obtained from a tournament by deleting the edges of 3 disjoint stars whose centers form a directed triangle. If $\delta^+_\Delta > 0$ and $\delta^-_\Delta > 0$ and D does not have any sink then it has at least 2 vertices with SNP.*

Proof. **Claim 1:** *For every $a \in A$, $b \in B$ and $c \in C$ we have:*
$b \to x \to c \to y \to a \to z \to b$.

proof-claim 1: This is clear, due to the claim in the previous proof and the condition $\delta^+_\Delta > 0$ and $\delta^-_\Delta > 0$.

Claim 2: *If $K = V(D)$ then D has at least 3 vertices with the SNP.*
proof-claim 2: Let $H = D - \{x, y, z\}$. H is a tournament with no sink (dominated vertex). Then H has 2 vertices u and v with SNP in H. Without loss of generality we may assume that $u \in A$. But $y \to u \to z$, the adding the vertices x, y and z makes u gains only one vertex to its first out-neighborhood and x to its second out-neighborhood. Thus, also u has the SNP in D. Similarly, v has the SNP in D. Suppose, without loss of generality, that $|A| \geq |C|$. We have $C \cup \{y\} = N^+(x)$ and $A \cup \{z\} = N^{++}(x)$. Hence, $d^+(x) = |C| + 1 \leq |A| + 1 \leq d^{++}(x)$, whence, x has the SNP in D.

Claim 3: *D is a good digraph.*
proof-claim 3: Let $\mathcal{I}_\mathcal{D}$ be the interval graph of D. Let C_1 and C_2 be two distinct connected components of Δ. The three centers of the missing disjoint stars appear in each of the these two connected components, whence $K(C_1) \cap K(C_2) \neq \phi$. Therefore, $\mathcal{I}_\mathcal{D}$ is a complete graph, having only one connected component ξ. Then, $K = K(\xi)$. So if Δ is composed of non trivial strongly connected components, the result holds by lemma 3. Due to the condition $\delta^+_\Delta > 0$ and $\delta^-_\Delta > 0$, Δ has a non trivial strongly connected component C. Since x, y and z appear in C we have $N^+(x)\backslash K = N^+(y)\backslash K = N^+(z)\backslash K$. Now let $v \in K$. If v appears in a non trivial strongly connected component of Δ then $N^+(v)\backslash K = N^+(x)\backslash K = N^+(y)\backslash K = N^+(z)\backslash K$. Otherwise, due to the condition $\delta^+_\Delta > 0$ and $\delta^-_\Delta > 0$, v appears in a directed path P of Δ joining two non trivial strongly connected components of Δ. By the definition of losing relations we can prove easily that for all $a \in K(C_1)$, $b \in K(P)$ and $c \in K(C_2)$ we have $N^+(a)\backslash K(\xi) \subseteq N^+(b)\backslash K(\xi) \subseteq N^+(c)\backslash K(\xi)$. In particular, for $a = x = c$ and $b = v$, So the result follows.

To conclude, we apply the same argument of the proof of theorem 7. $\square$

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References

[1] N. Dean and B. J. Latka, *Squaring the tournament: an open problem*, Congress Numerantium 109 (1995) 37-80.

[2] D. Fisher, *Squaring a tournament: a proof of Dean's conjecture*, J. Graph Theory 23 (1996), 43-48.

[3] F. Havet and S. Thomaseé, *Median Orders of Tournaments: A Tool for the Second Neighborhood Problem and Sumner's Conjecture*, J. Graph Theory 35 (2000), 244-256.

[4] D. Fidler and R. Yuster, *Remarks on the second neighborhood problem*, J. Graph Theory 55 (2007), 208-220.

[5] S. Ghazal, *Seymour's second neighborhood conjecture for tournaments missing a generalized star*, J. Graph Theory (accepted, 24 June 2011).